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COMPARISON OF JACOBI-INTEGRABLE CASES OF THE PLANE AND THREE-DIMENSIONAL MOTION OF A BODY IN A MEDIUM IN THE CASE OF JET FLOW†

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The complete integrability of the plane problem of the motion of a rigid body in a resisting medium under jet flow conditions is shown, when one first integral, which is transcendental function of the quasi-velocities (in the sense of the theory of functions of a complex variable having essentially singular points), exists in the system of dynamic equations. It is assumed that all the interaction of the medium with the body is concentrated in that part of the surface of the body which has the form of a (one-dimensional) plate. The plane problem is extended to the three-dimensional case; then a complete set of first integrals exists in the system of dynamic equations: one analytic, one meromorphic and one transcendental. It is assumed here that all the interaction of the medium with the body is concentrated in that part of the surface of the body which has the form of a plane (two-dimensional) disc. An attempt is also made to construct an extension of the "low-dimensional" cases to the dynamics of a so-called four-dimensional rigid body, interacting with a medium which is concentrated in that part of the (three-dimensional) surface of the body which has the form of a (three-dimensional) sphere. The vector of the angular velocity of the motion of such a body in this case is six-dimensional, while the velocity of the centre of mass is four-dimensional.

1. FUNDAMENTAL HYPOTHESES AND FORMULATION OF THE PROBLEM

According to the hypothesis of quasi-stationarity [1, 2], in the interaction of a body with a medium under jet-flow conditions the generalized forces depend only on the generalized coordinates and the generalized velocities. Below we distinguish a class of problems for which the generalized forces depend only on the generalized velocities.

We will assume that a homogeneous rigid body of mass m freely performs plane-parallel motion in a medium which is at rest at infinity, and that a certain part of the outer surface of the body is a finite plane region (a plate) P, under conditions of jet flow of the medium [1, 2], and perpendicular to the plane of motion $O\xi\eta$, which contains the centre of mass C of a body. The region P intersects the plane $O\xi\eta$ along a section AB of length Δ (Fig. 1). The remaining part of the body surface is arranged inside the volume, bounded by the surface of the liquid, which is separated from the edge of the plate, and does not experience the action of the medium. Such conditions may arise, for example, after a body enters water [3, 4]. Hence, we will assume that all the interaction of the medium with the body is concentrated on that part of the surface of the body which has the form of a (one-dimensional) plate.

In view of the properties of jet flow, the action of the medium on the plate reduces to a force $\mathbf{S}$, the line of action of which is orthogonal to the region P, i.e. $\mathbf{S}$ does not change its direction with respect to the body.

We will connect a right system of coordinates $Dxyz$ to the body, the z axis of which moves parallel to itself, and we will assume, for simplicity, that Dzx is the plane of geometrical symmetry of the body.

We will consider the mode of rectilinear progressive retardation. This mode is possible when the following two conditions are satisfied: (1) the velocity of the body is orthogonal to the section AB, and (2) the perpendicular, dropped from the centre of gravity C of the body to the plane of the plate, pertains to the line of action of the force $\mathbf{S}$.

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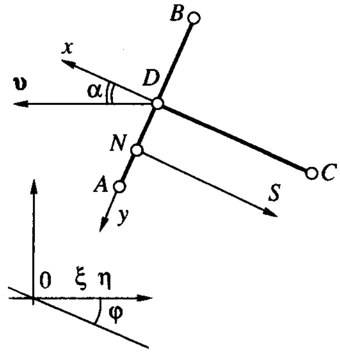

Fig. 1

When this mode is disturbed, the velocity vector of the point D, generally speaking. deviates from the straight line Dx by an angle of attack α. Then the point N of application of the force $\mathbf{S}$ is shifted along the plate AB by an amount y_N. The position of the body on the plane will be defined by the generalized coordinates $D = (\xi, \eta)$ and the angle φ between the straight line CD and the ξ axis.

We can now write the equations of motion of the centre of mass of the body in a projection onto the ξ and η axes, and the equation of the change in the angular momentum about the Koenig axis.

Introducing the quasi-velocities $|\upsilon| = \upsilon$, α instead of the velocities $\xi^\cdot$, $\eta^\cdot$ and the angular velocity Ω, as given by the formulae

$$\xi^\cdot = \upsilon\cos(\alpha+\varphi), \quad \eta^\cdot = \upsilon\sin(\alpha+\varphi), \quad \varphi^\cdot = \Omega \tag{1.1}$$

and assuming that

$$S = s_1\upsilon^2, \quad s_1 = s_1(\alpha, \omega) \geq 0, \quad y_N = y_N(\alpha, \omega), \quad \omega = \Omega\Delta/\upsilon$$

the equations of motion of the centre of mass in projections onto the Dx and Dy axes and the equation of the change in angular momentum can be written in the following form (compare with the equations proposed previously in [5])

$$\begin{aligned}
&\upsilon^\cdot\cos\alpha - \alpha^\cdot\upsilon\sin\alpha - \Omega\upsilon\sin\alpha + \sigma\Omega^2 = -s_1(\alpha, \omega)\upsilon^2 m^{-1}\operatorname{sign}\cos\alpha \\
&\upsilon^\cdot\sin\alpha + \alpha^\cdot\upsilon\cos\alpha + \Omega\upsilon\cos\alpha - \sigma\Omega^\cdot = 0 \\
&I\Omega^\cdot = y_N(\alpha, \omega)s_1(\alpha, \omega)\upsilon^2\operatorname{sign}\cos\alpha
\end{aligned} \tag{1.2}$$

($\sigma = CD$ and I is the central moment of inertia).

The system obtained is also regarded as an independent third-order system, and to obtain the trajectories of the rigid body in the plane it is necessary to supplement this system by the kinematic relations (1.1).

2. CLASSES OF DYNAMIC FUNCTIONS

It is difficult to obtain an explicit form of the pair of functions (y_N, s_1) for each specific body. Hence, it is sufficient to extend the class of functions $\{y_N\}$ and $\{s_1\}$, so that they necessarily include pairs of "real" functions. To do this it is necessary to extend the functions y_N and s_1 to finite angles of attack, i.e. "broaden" the regions in which this pair of dynamic functions is defined in the interval $(0, \pi/2)$. However, in fact, one must consider them over the whole number axis, as was done by Chaplygin [6], which, for an infinite strip, enables one to obtain these functions analytically.

We will consider the case when the pair of dynamic functions (y_N, s_1) depends only on the angle of attack (i.e. $y_N = y_N(\alpha)$, $s_1 = s_1(\alpha)$), in which case, to describe it qualitatively, one uses experimental data on the properties of jet flow [7].† We will introduce a sign-variable auxiliary function $s(\alpha) =$

† See also: SAMSONOV, V. A. and SHAMOLIN, M. V., A model problem on the motion of a body in a medium with a jet flow. Report of the Institute of Mechanics, Moscow State University, No. 3969, 1990.

$s_1(\alpha)$ sign cos α, which takes into account the sign of the projection of drag force onto the Dx axis. The classes of functions introduced are fairly wide and consist of fairly smooth, 2π-periodic functions of the following form: $y_N(\alpha)$ is an odd function and $s(\alpha)$ is an even function, which satisfy the following conditions: $y_N(\alpha) > 0$ when $\alpha \in (0, \pi)$, and $y_N'(0) > 0$, $y_N'(\pi) < 0$ (the class of functions $\{y_N\} = Y$); $s(\alpha) > 0$ when $\alpha \in (0, \pi/2)$ and $s(\alpha) < 0$ when $\alpha \in (\pi/2, \pi)$, when $s(0) > 0$, $s'(\pi/2) < 0$ (the class of functions $\{s\} = \Sigma$). Both functions change sign when α is replaced by $\alpha + \pi$. N particular, the following analytical functions (corresponding exactly to the Chaplygin case [6]) serve as typical representatives of the classes described above

$$y_{N0}(\alpha) = A\sin\alpha \in Y, \quad s_0(\alpha) = B\cos\alpha \in \Sigma; \quad A, B > 0 \tag{2.1}$$

3. MOTION WHEN THERE IS A FOLLOWER FORCE AND A SYSTEM WITH VARIABLE DISSIPATION "WITH ZERO MEAN"

We will distinguish further a class of problems on the motion of a body in a medium when, in addition to the action of the medium on the body, a certain follower force (a thrust) T is applied to the body along the straight line DC. One of these problems was solved in [8] for a constant thrust.

We will assume that the following equality is satisfied at all instants of time

$$v = \text{const} \tag{3.1}$$

This is possible if

$$|\mathbf{T}| = T = m\sigma\Omega^2 + v^2[s(\alpha) - m\sigma I^{-1}F(\alpha)\operatorname{tg}\alpha]$$

System (1.2), under certain conditions [5], can then be reduced to the system

$$\begin{aligned} &\alpha^{\cdot} = -\Omega + A_1F(\alpha)/\cos\alpha, \quad \Omega^{\cdot} = A_2F(\alpha); \quad A_1 = \sigma v/I, \quad A_2 = v^2/I, \\ &F(\alpha) = y_N(\alpha)s(\alpha) \end{aligned} \tag{3.2}$$

which is equivalent to the equation of a non-linear oscillator

$$\alpha^{\cdot\cdot} - A_1\alpha^{\cdot}F(\alpha)/\cos\alpha + A_2F(\alpha) = 0 \tag{3.3}$$

It can be seen that the motion of the system occurs due to the action of two forces: a conservative force $A_2F(\alpha)$ and a force that is linear in the velocity or with a variable coefficient $d(F(\alpha)/\cos\alpha/d\alpha$, which changes sign on transferring from one strip of the phase cylinder to another. We therefore have a system with a so-called variable dissipation (in the sense of the sign). It is obvious that in this case the dissipation vanishes on average over a period of 2π of the angle of attack. Hence, we will say that system (3.2) (or Eq. (3.3)) is a system (or an equation) with variable dissipation with zero mean.

4. THE INTEGRABILITY OF SYSTEMS IN TRANSCENDENTAL FUNCTIONS

We will make an assertion connecting the behaviour of the trajectories near asymptotic limiting sets and the integrability of the system. Repulsive and attractive limiting set will be called asymptotic limiting sets.

Consider a system of equations in the phase space R^n.

Theorem 1. If a system possesses asymptotic limiting sets, it does not have a complete set of continuous first integrals in the whole of phase space.

This theorem, although it can be proved fairly simply, has an important topological meaning, which touches on the continuity of the first integrals near limiting sets.

Corollary. From the point of view of the theory of functions, the first integrals can have essentially singular points in these asymptotic limiting sets.

System (3.2) has equilibrium positions $(\pi k, 0)$ $(k \in Z)$, which are repulsive when $k = 2n$ $(n \in Z)$ and are attractive when $k = 2n + 1$.

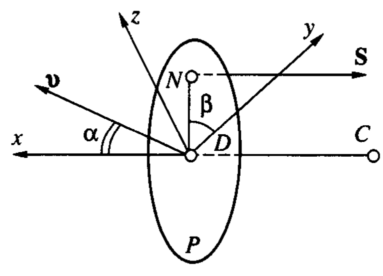

Fig. 2

Proposition. System (3.2) possesses a transcendental first integral [5, 8].

The required first integral in case (2.1) has the following form (three cases are possible depending on the sign of $\Delta = A_1^2 - 4A_2$):

$$\Delta < 0{:}\ [\Omega^2 + A_1\Omega\sin\alpha + A_2\sin^2\alpha]\sin\left\{\frac{2A_1}{\sqrt{-\Delta}}\operatorname{arctg}\frac{2\Omega + A_1\sin\alpha}{\sqrt{-\Delta}}\right\} = \text{const}$$

$$\Delta > 0{:}\ \chi_+\chi_+ = \text{const}\left(\chi_\pm = \left|2\Omega + A_1 \pm \sqrt{\Delta}\sin\alpha\right|^{\sqrt{\Delta} \mp A_1}\right)$$

$$\Delta = 0{:}\ |2\Omega + A_1\sin\alpha|\exp\left(-\frac{A_1\sin\alpha}{2\Omega + A_1\sin\alpha}\right) = \text{const}$$

Corollary. When $A_1 = 0$ the transcendental first integral (transcendental in the sense of the theory of functions of a complex variable), is converted into an analytical first integral for the standard equation of a mathematical pendulum.

5. FORMULATION OF THE THREE-DIMENSIONAL PROBLEM OF THE MOTION OF A BODY IN A RESISTING MEDIUM UNDER QUASI-STATIONARITY CONDITIONS

The dynamic model of the interaction of a rigid body with a resisting medium in the case of jet flow [1, 2] under quasi-stationarity conditions considered below not only enables one to extend the results of the corresponding problem of the plane-parallel motion of a body, interacting with the medium [3–8], and to obtain their three-dimensional analogues, but also enables one to obtain new cases that are Jacobi integrable. Sometimes the integrals are expressed in terms of elementary functions. Below we will show the integrability of the problem of the three-dimensional motion of a body when there is a servo constraint of the form (3.1)

We will assume that a homogeneous axisymmetrical rigid body of mass m performs three-dimensional motion in a medium and that a certain part of the body surface is in the form of a plane disc, which is under conditions of jet flow of the medium. The remaining part of the body surface does not interact with the medium. As in the case of plane-parallel motion, the force **S** with which the medium acts on the body is orthogonal to the plane of the disc. All similar assumptions of a model form in this case are real and are transferred from plane dynamics (Fig. 2). hence, it is assumed that all the interaction of the medium with the body is concentrated on that part of the body surface which has the form of a plane (two-dimensional) disc.

Further, we mention the changes in the formulation which are characteristic for the three-dimensional case. If we contact a system of coordinates $Dxyz$ with the body so that the Dx axis is directed along the axis of geometrical symmetry of the body, while the Dy and Dz axes lie in the plane of the disc, the position of the body in space will be defined by three Cartesian coordinate ξ, η, ζ of the point D and three angles θ, ψ, φ. In dynamical space we now have as the quasi-velocities the quantities (υ, α, β) – the spherical coordinates of the vector **v** of the velocity of the point D (α is the angle of attack, and the angle β is measured in the plane of the disc), and also (Ω_x, Ω_y, Ω_z) are the projections of the angular velocity on the associated axes $Dxyz$.

The system of dynamic equations in six-dimensional dynamic space has the form

$$
\begin{aligned}
&v^{\cdot}\cos\alpha - \alpha^{\cdot}v\sin\alpha + \Omega_y v\sin\alpha\sin\beta - \Omega_z v\sin\alpha\cos\beta + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)m^{-1}v^2 \\
&v^{\cdot}\sin\alpha\cos\beta + \alpha^{\cdot}v\cos\alpha\cos\beta - \beta^{\cdot}v\sin\alpha\sin\beta + \\
&+ \Omega_z v\cos\alpha - \Omega_x v\sin\alpha\sin\beta - \sigma\Omega_x\Omega_y - \sigma\Omega_z^{\cdot} = 0 \\
&v^{\cdot}\sin\alpha\sin\beta + \alpha^{\cdot}v\cos\alpha\sin\beta + \beta^{\cdot}v\sin\alpha\cos\beta + \\
&+ \Omega_x v\sin\alpha\cos\beta - \Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\Omega_y^{\cdot} = 0 \\
&I_1\Omega_x^{\cdot} + (I_3 - I_2)\Omega_y\Omega_z = 0, \quad I_2\Omega_y^{\cdot} + (I_1 - I_3)\Omega_x\Omega_z = -z_N s(\alpha)v^2, \\
&I_3\Omega_z^{\cdot} + (I_2 - I_1)\Omega_x\Omega_y = y_N s(\alpha)v^2
\end{aligned}
\tag{5.1}
$$

In addition to the function $s(\alpha)$ the system also contains the functions y_N and z_N ($0, y_N$ and z_N are the coordinates of the point N in the system $Dxyz$), defined in terms of the dynamic function R as follows:

$$y_N(\alpha, \beta) = R(\alpha)\cos\beta, \quad z_N(\alpha, \beta) = R(\alpha)\sin\beta$$

We will construct the three-dimensional version of the motion of the body in case (3.1) and we will investigate it under the conditions $R \in Y, s \in \Sigma$.

In a twelfth-order general dynamical system, by virtue of the cyclicity of these coordinates, splitting of the independent subsystem (5.1) in six-dimensional dynamic phase space of the quasi-velocities $T^2\{\alpha, \beta\} \times R_+^1\{v\} \times R^3\{\Omega_x, \Omega_y, \Omega_z\}$ occurs.

The moment of the drag force, as before, is represented in a form that is quadratic in the velocity: $M = F(\alpha)v^2$, in which the function $F(\alpha) = R(\alpha)s(\alpha)$ appears (see (3.2)), for a qualitative description of which we will use existing experimental information on the properties of jet flow.

6. A DYNAMICALLY SYMMETRICAL FREE RIGID BODY WITH A SERVO CONSTRAINT

The equations of motion of a dynamically symmetrical free rigid body ($I_2 = I_3$; I_1, I_2 and I_3 are the principal moment of inertia) in $Dxyz$ axes when there is a servo constraint of the form (3.1) [5] (for plane version of this problem see above) allow of the first integral $\Omega_x = \Omega_{x0}$ and have the following form (for simplicity we assume $\Omega_{x0} = 0$)

$$\alpha^{\cdot} = -z_- + I_2^{-1}F(\alpha)\sigma v/\cos\alpha, \quad z_-^{\cdot} = I_2^{-1}F(\alpha)v^2 - z_+^2\operatorname{ctg}\alpha, \quad z_+^{\cdot} = z_+z_-\operatorname{ctg}\alpha \tag{6.1}$$

$$\beta^{\cdot} = z_+\operatorname{ctg}\alpha \tag{6.2}$$

Here $\sigma = CD$, C is the centre of mass and $z_\pm = \Omega_y\cos\beta \pm \Omega_z\sin\beta$.

Assertion. Dynamical system (6.1), (6.2) is topologically equivalent to system (6.1), (6.2) with condition (2.1), i.e. when

$$F = F_0(\alpha) = F^0\sin\alpha\cos\alpha, \quad F^0 > 0 \tag{6.3}$$

When condition (6.3) is satisfied, Eq. (6.2) keeps its form, while system (6.1) takes the form

$$\alpha^{\cdot} = -z_- + \sigma n_0^2 v\sin\alpha, \; z_-^{\cdot} = n_0^2v^2\sin\alpha\cos\alpha - z_+^2\operatorname{ctg}\alpha, \; z_+^{\cdot} = z_+z_-\operatorname{ctg}\alpha \;\; (n_0^2 = F^0I_2^{-1}) \tag{6.4}$$

We will estimate the possibilities of complete integration of system (6.4), (6.2). Below we will present the first integrals of the system, which are expressed in terms of elementary functions.

Theorem 2. System (6.4) possesses a complete set of transcendental first integrals. System (6.4), (6.2) is also completely Jacobi integrable, and its first two integrals are the first integrals of system (6.4).

The meromorphic integral of system (6.4) has the form

$$(z_+^2 + z_-^2 - \sigma n_0^2 v z_- \tau + n_0^2 v^2 \tau^2)/z_+ \tau = C_1, \quad \tau = \sin\alpha \tag{6.5}$$

Since system (6.4) possess variable dissipation and is analytic, we can obtain two other additional integrals for it in explicit form. By virtue of Eq. (6.5) the following identity is satisfied

$$u_+ = (C_1 + G)/2; \quad G = \sqrt{C_1^2 - 4[u_-^2 - \sigma n_0^2 v u_- + n_0^2 v^2]}, \quad u_\pm = z_\pm \tau \tag{6.6}$$

Then, in view of Eqs (6.4) and (6.6) the quadrature for finding the desired integral, which connects the quantities u_- and τ, takes the form

$$\int\frac{d\tau}{\tau} = \frac{\sigma n_0^2 v}{2} D_1 - D_2, \quad D_1 = \pm\int\frac{dy}{(y + C_1)\sqrt{C_1^2 - y^2 + 4a}}, \quad D_2 = -2\int\frac{dx}{y^2 \mp G} \tag{6.7}$$

$$y^2 = C_1^2 - 4(x - a), \quad x = (u_- - \sigma n_0^2/2)^2, \quad a = n_0^2 v^2 \Delta_2/4, \quad \Delta_2 = \sigma^2 n_0^2 - 4$$

Suppose, to be specific that $C_1^2 + 4a \geq 0$. Then

$$D_1 = \pm\frac{1}{2n_0 v\sqrt{-a}}\arcsin\frac{C_1 y + C_1^2 + 2n_0 v\sqrt{a}}{(y + C_1)\sqrt{C_1^2 + 2n_0 v\sqrt{a}}} + C_2, \quad \text{if} \quad \Delta_2 < 0$$

$$D_1 = \mp(C_1 y + C_1^2)^{-1}\sqrt{C_1^2 - y^2} + C_2, \quad \text{if} \quad \Delta_2 = 0$$

$$D_1 = \mp(\zeta_+ + \zeta_-) + C_2\left(\zeta_+ = \frac{1}{4n_0 v\sqrt{a}}\ln\left|\frac{2\sqrt{a} \pm G_1}{y + C_1} + \frac{C_1}{2\sqrt{a}}\right|\right), \quad \text{if} \quad \Delta_2 > 0$$

Reverting to the old variables, it can be shown that the additional first integral has the following structural form (analogous to the transcendental first integral corresponding to the case of the plane-parallel motion of the body (compare with the case considered earlier in [5]))

$$\ln|\sin\alpha| + G_2(z_-\sin\alpha, z_+\sin\alpha, \sin\alpha) = C_2$$

The additional integral of the system obtain above, which is transcendental function of the phase variables, is, together with the integral (6.5), a complete set of first integrals of system (6.4). For the complete system (6.4), (6.2) with $\Omega_{x0} = 0$, one more first integral is required.

To find the additional integral of system (6.4), (6.2) we note that since $dz_+/d\beta = z$, we have $du_+/d\beta + [-u_- + \sigma n_0^2 v] = u_-$. Hence

$$du_+/d\beta = \pm\sqrt{\sigma^2 n_0^4 v^2 - 4[u_+^2 - C_1 u_+ + n_0^2 v^2]}$$

and, consequently, the following equality is satisfied

$$\cos^2[2(\beta + C_3)] = G_3^{-1}(u_- - \sigma n_0^2/2)^2 u_+^2$$

where

$$G_3 = [u_+^2 - \sigma n_0^2 v u_+]^2 + 2[u_+^2 - \sigma n_0^2 v u_+][u_-^2 + n_0^2 v^2] + [u_-^2 + n_0^2 v^2]^2 + \sigma^2 n_0^4 v^2 u_-^2$$

7. THE TRANSITION TO THE FOUR-DIMENSIONAL CASE

The structure of the dynamic equations of motion are often preserved when the dynamic properties are transferred to the case of higher dimensions. For example, a theory of the motion of a four-dimensional (or even an n-dimensional) rigid body has recently been developed [9–1]; the Hamiltonian nature of the equations of motion of a multidimensional rigid body has been shown in certain cases. It is interesting to investigate the motion of a so-called four-dimensional rigid body, which interacts with a "resisting medium" in accordance with the laws of "jet flow". In the latter case it is assumed that all the interaction of the (four-dimensional) rigid body with the medium is concentrated on part of the (three-dimensional) surface of the body, which has the form of a (three-dimensional) sphere. Then the vector of the angular velocity of motion of such a body is six-dimensional, and the velocity of the centre of mass is four-dimensional.

Remark. Previously (in fundamental geometries) only those motions of a four-dimensional body were considered when the moment of the external forces is equal to zero. We are developing a new technique for investigating the equations of motion of a rigid body in the set $so(4) \times R^4$ when the moment of the external forces is non-zero.

The procedure for integrating the dynamical systems considered can nearly always be extended to $so(n) \times R^n$ space of the dynamic equations of an arbitrary dynamically symmetrical n-dimensional rigid body.

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